

COUNTING FUNDAMENTAL SOLUTIONS TO THE PELL EQUATION WITH PRESCRIBED SIZE

PING XI

ABSTRACT. The cardinality of the set of $D \leq x$ for which the fundamental solution of the Pell equation $t^2 - Du^2 = 1$ is less than $D^{\frac{1}{2}+\alpha}$ with $\alpha \in [\frac{1}{2}, 1]$ is studied and certain lower bounds are obtained, improving previous results by Fouvry.

1. INTRODUCTION

Let D be a non-square positive integer. The Pell equation is usually referred to

$$(1.1) \quad t^2 - Du^2 = 1,$$

to which the solution can be written in the usual form

$$\eta_D := t + u\sqrt{D}.$$

The classical Dirichlet Units Theorem asserts that the set of solutions to (1.1) is non-trivial and has the form

$$\{\eta_D : \eta_D \text{ solution of (1.1)}\} = \{\pm \varepsilon_D^n : n \in \mathbf{Z}\},$$

where ε_D is called the fundamental solution of (1.1) and is given by

$$\varepsilon_D := \inf\{\eta_D : \eta_D > 1\}.$$

Writing $\varepsilon_D := t_0 + u_0\sqrt{D}$, we have $t_0, u_0 \geq 1$, from which we deduce that $t_0 = \sqrt{1 + u_0^2 D} > \sqrt{D}$ and finally

$$\varepsilon_D > 2\sqrt{D}.$$

We are interested in counting the integers D for which ε_D or η_D is less than a fixed power of D .

For $\alpha > 0$ and $x \geq 2$, define

$$\begin{aligned} S(x, \alpha) &:= |\{(\eta_D, D) : 2 \leq D \leq x, D \neq \square, \eta_D \leq D^{\frac{1}{2}+\alpha}\}|, \\ S^f(x, \alpha) &:= |\{(\varepsilon_D, D) : 2 \leq D \leq x, D \neq \square, \varepsilon_D \leq D^{\frac{1}{2}+\alpha}\}|. \end{aligned}$$

In his pioneer work, Hooley [Ho82] proved the following theorem.

Theorem A (Hooley). *Let ε_0 satisfy $0 < \varepsilon_0 < \frac{1}{2}$. As $x \rightarrow +\infty$, one has*

$$S(x, \alpha) = S^f(x, \alpha) \sim \frac{4\alpha^2}{\pi^2} \sqrt{x} \log^2 x$$

uniformly for $\varepsilon_0 \leq \alpha \leq \frac{1}{2}$.

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In the same paper, Hooley [Ho82] also made the following conjecture.

Conjecture 1.1 (Hooley). *For any given $\alpha > \frac{1}{2}$, we have*

$$S(x, \alpha) \sim \frac{1}{\pi^2} \left(4\alpha - 1 + C(\alpha) \right) \sqrt{x} \log^2 x,$$

where

$$C(\alpha) = \begin{cases} 0, & \alpha \in (\frac{1}{2}, 1], \\ \frac{1}{18}(\alpha - 1)^2, & \alpha \in (1, \frac{5}{2}], \\ \frac{1}{24}(4\alpha - 7), & \alpha \in (\frac{5}{2}, +\infty). \end{cases}$$

Fouvry [Fo16] made a first significant step towards Hooley's conjecture in the case $\alpha \in (\frac{1}{2}, 1]$. In fact, he proved the following theorem.

Theorem B (Fouvry). *As $x \rightarrow +\infty$, one has*

$$(1.2) \quad S^f(x, \alpha) \geq \frac{1}{\pi^2} \left(4\alpha - 1 - 4 \left(\alpha - \frac{1}{2} \right)^2 - o(1) \right) \sqrt{x} \log^2 x$$

and

$$(1.3) \quad S(x, \alpha) \geq \frac{1}{\pi^2} \left(4\alpha - 1 - 3 \left(\alpha - \frac{1}{2} \right)^2 - o(1) \right) \sqrt{x} \log^2 x$$

uniformly for $\alpha \in [\frac{1}{2}, 1]$.

Moreover, Fouvry [Fo16] introduced a conjectural estimation for exponential sums.

Conjecture 1.2. *There exists an absolute $\vartheta \in [\frac{1}{2}, 1)$, such that for any integer $k \geq 0$, one has the inequality*

$$\sum_{\substack{N < n \leq N_1 \\ m \equiv a \pmod{4^k}, (m, n) = 1}} e\left(\frac{h\bar{n}^2}{m^2}\right) \ll_k (h, m^2)^{\frac{1}{2}} N^\vartheta$$

uniformly for any integers a, h, m satisfying $h \neq 0, m \geq 1$ and $2 \nmid am$, for any real number N satisfying $m \leq N \leq m^2$ and $N < N_1 \leq 2N$.

Assuming Conjecture 1.2, Fouvry [Fo16] derived the following stronger lower bounds.

Theorem C (Fouvry). *Assume that Conjecture 1.2 is true for some $\vartheta \in [\frac{1}{2}, 1)$. As $x \rightarrow +\infty$, one has*

$$S^f(x, \alpha) \geq \frac{1}{\pi^2} \left(4\alpha - 1 - o(1) \right) \sqrt{x} \log^2 x$$

and

$$S(x, \alpha) \geq \frac{1}{\pi^2} \left(4\alpha - 1 + \left(\alpha - \frac{1}{2} \right)^2 - o(1) \right) \sqrt{x} \log^2 x$$

uniformly for $\alpha \in [\frac{1}{2}, \frac{1}{1+\vartheta}]$.

The first inequality coincides with Hooley's Conjecture 1.1 when α is slightly larger than $\frac{1}{2}$. On the other hand, Bourgain [Bo14] considered Conjecture 1.2 itself. In particular, he succeeded in saving a power of $\log N$ in the trivial bound for the sum involved. This allows him to improve the lower bound (1.2) by replacing the term $-4(\alpha - \frac{1}{2})^2$ with the term $O((\alpha - \frac{1}{2})^{2+c})$, where c is some positive constant.

Bourgain's improvement is of interest only if α is quite close to $\frac{1}{2}$ and one should pay much more attention if every parameter is to be made effective. The aim of this paper is to give another improvement to Theorem B towards Conjecture 1.1.

Theorem 1.1. *Let $x \rightarrow +\infty$. For any fixed $\theta \in (0, \frac{1}{2})$, we have*

$$(1.4) \quad S^f(x, \alpha) \geq \frac{1}{\pi^2} \left\{ 4\alpha - 1 - 4\left(\alpha - \frac{1}{2}\right)^2 + \frac{1}{6}\rho\left(\frac{1}{\theta}\right)F_\theta(\alpha) - o(1) \right\} \sqrt{x} \log^2 x$$

and

$$(1.5) \quad S(x, \alpha) \geq \frac{1}{\pi^2} \left\{ 4\alpha - 1 - 3\left(\alpha - \frac{1}{2}\right)^2 + \frac{1}{6}\rho\left(\frac{1}{\theta}\right)F_\theta(\alpha) - o(1) \right\} \sqrt{x} \log^2 x$$

uniformly in $\alpha \in [\frac{1}{2}, 1]$, where ρ is the Dickman function given by (A.1) and

$$(1.6) \quad F_\theta(\alpha) = \begin{cases} 24\alpha - 4(5 + 2\theta)\alpha^2 - (7 - 2\theta), & \alpha \in [\frac{1}{2}, \frac{6}{11+2\theta}], \\ 864(11 + 2\theta)^{-2} - (7 - 2\theta), & \alpha \in (\frac{6}{11+2\theta}, 1]. \end{cases}$$

For any fixed $\theta \in (0, \frac{1}{2})$, one may check that $F_\theta(\alpha)$ is positive in $\alpha \in (\frac{1}{2}, 1]$ and increasing in $\alpha \in [\frac{1}{2}, \frac{6}{11+2\theta}]$. In particular, for $\alpha \in (\frac{1}{2}, \frac{6}{11})$, we take $\theta = \frac{3}{\alpha} - \frac{11}{2}$ such that $\alpha = \frac{6}{11+2\theta}$ and

$$\frac{1}{6}\rho\left(\frac{1}{\theta}\right)F_\theta(\alpha) = \rho\left(\frac{2\alpha}{6-11\alpha}\right) \frac{4(\alpha - \frac{1}{2})^3 + 6(\alpha - \frac{1}{2})^2}{\alpha}.$$

Moreover, $\rho(\frac{2\alpha}{6-11\alpha})/\alpha > 0.3008$ for $\alpha \in (\frac{1}{2}, \frac{35}{69}]$. Hence we may conclude the following consequence.

Corollary 1.1. *Let $x \rightarrow +\infty$. Uniformly for $\alpha \in (\frac{1}{2}, \frac{35}{69}]$, we have*

$$S^f(x, \alpha) \geq \frac{1}{\pi^2} \left\{ 4\alpha - 1 - \frac{11}{5}\left(\alpha - \frac{1}{2}\right)^2 + \frac{6}{5}\left(\alpha - \frac{1}{2}\right)^3 \right\} \sqrt{x} \log^2 x$$

and

$$S(x, \alpha) \geq \frac{1}{\pi^2} \left\{ 4\alpha - 1 - \frac{6}{5}\left(\alpha - \frac{1}{2}\right)^2 + \frac{6}{5}\left(\alpha - \frac{1}{2}\right)^3 \right\} \sqrt{x} \log^2 x.$$

The framework of the proof is based on [Fo16]. To make the paper clear, we will present the proof as complete as possible, but also with omitting some details that are not quite essential to understand the underlying ideas.

The key point of proving Theorem 1.1 is a variant of Conjecture 1.2 that can be proved unconditionally. More precisely, if one allows the moduli m^2 to be smooth numbers (integers free of large prime factors), it is possible to prove the existence of ϑ in Conjecture 1.2 if N is not too small. The details can be referred to Theorem 3.1 and Section 4. We will adopt the q -analogue of van der Corput method, which can be at least dated back to Heath-Brown [HB78] on the proof of Weyl-type subconvex bounds for Dirichlet L -functions to well-factorable moduli. Instead of the AB -process in [HB78], we apply the BAB -process by introducing a completion in the initial step. It is expected that one can do better on the exponential sums in Conjecture 1.2 if better factorizations of the moduli are imposed; see [WX16] for instance in the case of squarefree moduli. However, the improvements to Theorem 1.1 would be rather slight, since the density of smooth numbers decays rapidly when the size of their prime factors decreases.

As an extension to Theorem 1.1, one may consider

$$S^f(x; \alpha, \beta) := |\{(\varepsilon_D, D) : 2 \leq D \leq x, D \neq \square, D^{\frac{1}{2}+\beta} \leq \varepsilon_D \leq D^{\frac{1}{2}+\alpha}\}|$$

for $\alpha > \beta \geq 0$. Conjecture 1.1 would yield asymptotics for $S(x; \alpha, \beta)$ while α, β are of different prescribed sizes. A weaker statement would assert that, for any $\alpha > \beta \geq 0$, there exists a positive constant $c = c(\alpha, \beta)$, such that

$$S^f(x; \alpha, \beta) \geq c\sqrt{x} \log^2 x$$

for all large $x > x_0(\alpha, \beta)$. This weaker statement was made unconditionally by Fouvry and Jouve [FJ12] whenever $\beta < \frac{3}{2}$. It is expected that the arguments in this paper can enlarge the admissible range of β .

Notation and convention. As usual, $e(x) = e^{2\pi ix}$, φ denote the Euler function and $\omega(q)$ counts the number of distinct prime factors of q . The variable p is reserved for prime numbers. For a real number x , denote by $[x]$ its integral part and $\|x\| = \min_{n \in \mathbf{Z}} |x - n|$. From time to time, we use (m, n) to denote the greatest common divisor of m, n , and also to denote a tuple given by two coordinates; these will not cause confusions as one will see later. The symbol $*$ in summation reminds us to sum over primitive elements such that poles of the summand are avoided. For a function $f \in L^1(\mathbf{R})$, its Fourier transform is defined as

$$\widehat{f}(y) := \int_{\mathbf{R}} f(x) e(-yx) dx.$$

We use ε to denote a very small positive number, which might be different at each occurrence; we also write $X^\varepsilon \log X \ll X^\varepsilon$.

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2. FUNDAMENTAL TRANSFORMATIONS: AFTER HOOLEY AND FOUVRY

We first make some fundamental transformations following the arguments of Hooley and Fouvry. For some conclusions, we omit the proof and the detailed arguments can be found in [Ho82] and [Fo16].

2.1. An initial transformation. First, we write

$$(2.1) \quad S(x, \alpha) = \sum_{\substack{\square \neq D \leq x \\ t^2 - Du^2 = 1 \\ 1 < t + u\sqrt{D} \leq D^{\frac{1}{2} + \alpha}}} \sum_{t \geq 1} \sum_{u \geq 1} 1 = \sum_{1 \leq u \leq X_\alpha} \sum_{\substack{Y_2(u, \alpha) \leq t \leq Y_3(u) \\ Du^2 = t^2 - 1}} 1,$$

where

$$(2.2) \quad X_\alpha := \frac{1}{2}(x^\alpha - x^{-1-\alpha});$$

$$(2.3) \quad Y_3(u) := \sqrt{xu^2 + 1};$$

$$(2.4) \quad Y_2(u, \alpha) := \sqrt{Y_1(u, \alpha)u^2 + 1}.$$

Here, $Y_1(u, \alpha)$ is a function in u , implicitly defined by the equation

$$u = \frac{1}{2}(Y_1(u, \alpha)^\alpha - Y_1(u, \alpha)^{-1-\alpha}).$$

We have the following asymptotic characterization for $Y_2(u, \alpha)$. The proof can be found in [Fo16, Lemma 2.1] and the subsequent remark.

Lemma 2.1. *Let $\alpha > 0$. The function $u \mapsto Y_2(u, \alpha)$ is of \mathcal{C}^∞ -class and satisfies the inequalities*

$$(2.5) \quad 2^{\frac{1}{2\alpha}} u^{1+\frac{1}{2\alpha}} < Y_2(u, \alpha) < (2^{\frac{1}{2\alpha}} + o(1)) u^{1+\frac{1}{2\alpha}}$$

and

$$(2.6) \quad \frac{d}{du} Y_2(u, \alpha) = O(u^{\frac{1}{2\alpha}})$$

as $u \rightarrow +\infty$.

2.2. A first dissection of $S(x, \alpha)$. We truncate the u -sum in (2.1) at $X_{\frac{1}{2}}$, and the contributions from $u \leq X_{\frac{1}{2}}$ and $u > X_{\frac{1}{2}}$ are respectively denoted by $A(x, \alpha)$ and $B(x, \alpha)$. Therefore,

$$(2.7) \quad S(x, \alpha) = A(x, \alpha) + B(x, \alpha).$$

Accordingly, we may define $A^f(s, \alpha)$ and $B^f(x, \alpha)$ by introducing the extra restriction $t + u\sqrt{D} = \varepsilon_D$ to the equation $t^2 - Du^2 = 1$.

As stated by Fouvry [Fo16, Formula (4.5)], we have

Lemma 2.2. *Let $\alpha \in [\frac{1}{2}, 1]$. As $x \rightarrow +\infty$, we have*

$$(2.8) \quad A(x, \alpha) \sim \frac{1}{\pi^2} \sqrt{x} \log^2 x.$$

Our task thus reduces to proving a lower bound for $B(x, \alpha)$. Put $\mathcal{R}(u) := \{\Omega \pmod{u^2} : \Omega^2 \equiv 1 \pmod{u^2}\}$. We then have

$$(2.9) \quad B(x, \alpha) = \sum_{X_{\frac{1}{2}} < u \leq X_\alpha} \sum_{\Omega \in \mathcal{R}(u)} \sum_{\substack{Y_2(u, \alpha) \leq t \leq Y_3(u) \\ t \equiv \Omega \pmod{u^2}}} 1.$$

Put $\gamma(u) = |\mathcal{R}(u)|$. Thus $u \mapsto \gamma(u)$ is multiplicative and satisfies

$$(2.10) \quad \begin{cases} \gamma(2) = 2, \gamma(2^k) = 4 \text{ for } k \geq 2, \\ \gamma(p^\ell) = 2 \text{ for } p \geq 3, \ell \geq 1. \end{cases}$$

2.3. Analysis of $\mathcal{R}(u)$. For $u \geq 1$, write $u = 2^k u_0$, where u_0 is an odd integer. The choice of (k, u_0) is unique for each $u \geq 1$. The Chinese remainder theorem implies

$$(2.11) \quad \Omega^2 \equiv 1 \pmod{u^2} \iff \begin{cases} \Omega^2 \equiv 1 \pmod{u_0^2}, \\ \Omega^2 \equiv 1 \pmod{4^k}. \end{cases}$$

In this way, one can establish a bijection between $\mathcal{R}(u)$ and $\mathcal{R}(2^k) \times \mathcal{R}(u_0)$. Starting from (2.9), we decompose $B(x, \alpha)$ by

$$(2.12) \quad B(x, \alpha) = \sum_{k \geq 0} \sum_{\xi \in \mathcal{R}(2^k)} B(x, \alpha; \xi, k),$$

where

$$B(x, \alpha; \xi, k) = \sum_{\substack{2^{-k} X_{\frac{1}{2}} < u \leq 2^{-k} X_\alpha \\ 2 \nmid u}} \sum_{\Omega \in \mathcal{R}(u)} \sum_{\substack{Y_2(2^k u, \alpha) \leq t \leq Y_3(2^k u) \\ t \equiv \Omega \pmod{u^2} \\ t \equiv \xi \pmod{4^k}}} 1.$$

The task will be evaluating $B(x, \alpha; \xi, k)$ for all $k \geq 0$ and $\xi \in \mathcal{R}(2^k)$. This would require the following description of $\mathcal{R}(u)$ for odd u . This is Lemma 4.1 in [Fo16].

Lemma 2.3. *Let u be a positive odd integer. Then there is a bijection Φ between the set of coprime decompositions of u*

$$\mathcal{D}(u) := \{(u_1, u_2) : u_1 u_2 = u, (u_1, u_2) = 1, u_1, u_2 \geq 1\}$$

and the set of roots of congruence

$$\mathcal{R}(u) := \{\Omega \pmod{u^2} : \Omega^2 \equiv 1 \pmod{u^2}\}.$$

Such a bijection can be defined by $\Phi(u_1, u_2) = \Omega$, where Ω is uniquely determined by the congruences $\Omega \equiv 1 \pmod{u_1^2}$ and $\Omega \equiv -1 \pmod{u_2^2}$. In an equivalent manner, we have the congruences

$$\Phi(u_1, u_2) \equiv -\bar{u}_1^2 u_1^2 + \bar{u}_2^2 u_2^2 \pmod{u^2}.$$

Here $\bar{u}_1 u_1 \equiv 1 \pmod{u_2}$ and $\bar{u}_2 u_2 \equiv 1 \pmod{u_1}$.

With the help of Lemma 2.3, we may rewrite $B(x, \alpha; k, \xi)$ as

$$\begin{aligned} B(x, \alpha; \xi, k) &= \sum_{\substack{2^{-k} X_{\frac{1}{2}} < u_1 u_2 \leq 2^{-k} X_{\alpha} \\ (2, u_1 u_2) = (u_1, u_2) = 1}} \sum_{\substack{Y_2(2^k u_1 u_2, \alpha) \leq t \leq Y_3(2^k u_1 u_2) \\ t \equiv \Phi(u_1, u_2) \pmod{u_1^2 u_2^2} \\ t \equiv \xi \pmod{4^k}}} 1 \\ &=: B^>(x, \alpha; \xi, k) + B^<(x, \alpha; \xi, k), \end{aligned}$$

where $B^>(x, \alpha; \xi, k)$ and $B^<(x, \alpha; \xi, k)$ correspond to the restrictions $u_1 > u_2$ and $u_1 < u_2$, respectively. Since the treatments of $B^>(x, \alpha; \xi, k)$ and $B^<(x, \alpha; \xi, k)$ are similar, it suffices to study $B^<(x, \alpha; \xi, k)$ as presented in the next section.

We close this section by the following trivial equality:

$$(2.13) \quad S(x, \alpha) = S^f(x, \alpha) + S\left(x, \frac{\alpha}{2} - \frac{1}{4}\right), \quad \frac{1}{2} < \alpha \leq 1,$$

which is a consequence of the equivalence

$$\varepsilon_D^2 \leq D^{\frac{1}{2} + \alpha} \iff \varepsilon_D \leq D^{\frac{1}{2} + (\frac{\alpha}{2} - \frac{1}{4})}.$$

This allows us to transfer between $S^f(x, \alpha)$ and $S(x, \alpha)$.

3. LOWER BOUND FOR $B(x, \alpha)$

In order to conclude the lower bound for $B(x, \alpha)$, we now start the study of $B^<(x, \alpha; \xi, k)$. Recall that

$$B^<(x, \alpha; \xi, k) = \sum_{\substack{2^{-k} X_{\frac{1}{2}} < u_1 u_2 \leq 2^{-k} X_{\alpha} \\ u_1 < u_2 \\ (2, u_1 u_2) = (u_1, u_2) = 1}} \sum_{\substack{Y_2(2^k u_1 u_2, \alpha) \leq t \leq Y_3(2^k u_1 u_2) \\ t \equiv \Phi(u_1, u_2) \pmod{u_1^2 u_2^2} \\ t \equiv \xi \pmod{4^k}}} 1.$$

We would like to drop the multiplicative constraints $2^{-k} X_{\frac{1}{2}} < u_1 u_2 \leq 2^{-k} X_{\alpha}$ and sum over u_1, u_2 separately. To do so, we may introduce the following inequality

$$(3.1) \quad B^<(x, \alpha; \xi, k) \geq \sum_{\mathbf{U}} \sum_{\xi_1} \sum_{\xi_2} B(x, \alpha; \mathbf{U}, \xi, \xi_1, \xi_2, k),$$

where

- the summation is over all $\mathbf{U} = (U_1, U_2)$ satisfying

$$U_1 < U_2, \quad X_{\frac{1}{2}} < 2^k U_1 U_2 \leq \frac{X_\alpha}{8},$$

- the summation is over all $\xi_1, \xi_2 \pmod{4^k}$ satisfying $(\xi_1 \xi_2, 4^k) = 1$,
- we have defined

$$B(x, \alpha; \mathbf{U}, \xi, \xi_1, \xi_2, k) := \sum_{\substack{u_1 \sim U_1, u_2 \sim U_2 \\ u_1 \equiv \xi_1, u_2 \equiv \xi_2 \pmod{4^k} \\ (2, u_1 u_2) = (u_1, u_2) = 1}} \sum_{\substack{Y_2(2^k u_1 u_2, \alpha) \leq t \leq Y_3(2^k u_1 u_2) \\ t \equiv \Phi(u_1, u_2) \pmod{u_1^2 u_2^2} \\ t \equiv \xi \pmod{4^k}}} 1. \quad (1)$$

Of course the condition $(2, u_1 u_2) = 1$ can be dropped when $k \geq 1$. The parameter α is supposed to be fixed and the congruence conditions modulo 4^k are harmless. So to shorten the notations, we write $B(x, \mathbf{U}) := B(x, \alpha; \mathbf{U}, \xi, \xi_1, \xi_2, k)$. The coprimality condition $(u_1, u_2) = 1$ will be omitted, but kept in mind in all the computations below. Finally we shall not precise the dependence on k of some O -symbols, since we shall work with a finite number of values of k . The case $k = 0$ is typical and really reflects the difficulties of the method.

3.1. Reduction to exponential sums: after Fouvry. The congruence condition $t \equiv \Phi(u_1, u_2) \pmod{u_1^2 u_2^2}$ implies that $t \equiv -1 \pmod{u_2^2}$, i.e., $t = -1 + \ell u_2^2$ for some $\ell \in \mathbf{Z}$. Since $Y_2(2^k u_1 u_2, \alpha) \leq t \leq Y_3(2^k u_1 u_2)$, then there is no such t if u_2 is too large, for instance when

$$u_2^2 > \sqrt{4^k x u_1^2 u_2^2} + 1 + 1.$$

Hence we can suppose

$$(3.2) \quad U_2 \leq 2^{k+2} \sqrt{x} U_1,$$

otherwise $B(x, \mathbf{U}) = 0$.

Since $u_1 u_2$ is odd, we deduce from (??) the equivalence

$$t \equiv \Phi(u_1, u_2) \pmod{u_1^2 u_2^2}, \quad t \equiv \xi \pmod{4^k} \iff t \equiv t_0 \pmod{4^k u_1^2 u_2^2}$$

with

$$t_0 \equiv \xi u_1^2 u_2^2 \cdot \overline{(u_1^2 u_2^2)} + (2u_2^2 \overline{u_2^2} - 1) 4^k \overline{4^k} \pmod{4^k u_1^2 u_2^2},$$

where $u_1^2 u_2^2 \cdot \overline{(u_1^2 u_2^2)} \equiv 1 \pmod{4^k}$, $u_2^2 \overline{u_2^2} \equiv 1 \pmod{u_1^2}$ and $4^k \overline{4^k} \equiv 1 \pmod{u_1^2 u_2^2}$. It follows that

$$\frac{t_0}{4^k u_1^2 u_2^2} \equiv \kappa - \frac{1}{4^k u_1^2 u_2^2} + 2 \frac{\overline{4^k u_2^2}}{u_1^2} \pmod{1}$$

with $\kappa := (\xi + 1) \overline{\xi_1^2 \xi_2^2} / 4^k$. The three terms on the RHS have completely different structures: the first one is constant, the second one changes very slowly when u_1 and u_2 vary, the third one oscillates a lot when u_2 varies with u_1 fixed.

For each fixed k , we consider the sum

$$B(x, \mathbf{U}) = B(x, \alpha; \mathbf{U}, \xi, \xi_1, \xi_2, k) = \sum_{\substack{u_1 \sim U_1, u_2 \sim U_2 \\ u_1 \equiv \xi_1, u_2 \equiv \xi_2 \pmod{4^k} \\ (2, u_1 u_2) = (u_1, u_2) = 1}} \sum_{\substack{Y_2 \leq t \leq Y_3 \\ t \equiv t_0 \pmod{4^k u_1^2 u_2^2}}} 1,$$

where $Y_2 := Y_2(2^k u_1 u_2, \alpha)$ and $Y_3 := Y_3(2^k u_1 u_2)$. As in [Fol16], we smooth the t -sum via the following lemma.

Lemma 3.1. *For every $\delta > 0$ there exists a smooth function $g : \mathbf{R} \rightarrow \mathbf{R}$ which has the two properties*

$$0 \leq g \leq \mathbf{1}_{[-\frac{1}{2}, \frac{1}{2}]}$$

and

$$\int_{\mathbf{R}} g(y) dy = 1 - \delta.$$

Let g be a smooth function given as in Lemma 3.1. Hence

$$(3.3) \quad B(x, \mathbf{U}) \geq \sum_{\substack{u_1 \sim U_1, u_2 \sim U_2 \\ u_1 \equiv \xi_1, u_2 \equiv \xi_2 \pmod{4^k} \\ (2, u_1 u_2) = (u_1, u_2) = 1}} \sum_{\substack{t \in \mathbf{Z} \\ t \equiv t_0 \pmod{4^k u_1^2 u_2^2}}} g\left(\frac{t - \frac{Y_2 + Y_3}{2}}{Y_3 - Y_2}\right).$$

By Poisson summation, the t -sum becomes

$$\frac{Y_3 - Y_2}{4^k u_1^2 u_2^2} \sum_{h \in \mathbf{Z}} \widehat{g}\left(\frac{h(Y_3 - Y_2)}{4^k u_1^2 u_2^2}\right) e\left(h\kappa + 2h \frac{\overline{4^k u_2^2}}{u_1^2} - \frac{h(2 + Y_2 + Y_3)}{2 \cdot 4^k u_1^2 u_2^2}\right).$$

From integration by parts, we have $\widehat{g}(y) \ll (1 + |y|)^{-A}$ for any $A \geq 0$. Note that

$$\frac{Y_3 - Y_2}{4^k u_1^2 u_2^2} \asymp \frac{\sqrt{x}}{U_1 U_2}.$$

The above sum over h can be truncated to $0 \leq |h| \leq H$ with $H = U_1 U_2 x^{-\frac{1}{2} + \varepsilon}$, and the remaining contribution is at most $O(x^{-2017})$. Therefore,

$$\begin{aligned} B(x, \mathbf{U}) &\geq \sum_{\substack{u_1 \sim U_1, u_2 \sim U_2 \\ u_1 \equiv \xi_1, u_2 \equiv \xi_2 \pmod{4^k} \\ (2, u_1 u_2) = (u_1, u_2) = 1}} \frac{Y_3 - Y_2}{4^k u_1^2 u_2^2} \sum_{0 \leq |h| \leq H} \widehat{g}\left(\frac{h(Y_3 - Y_2)}{4^k u_1^2 u_2^2}\right) e\left(h\kappa + 2h \frac{\overline{4^k u_2^2}}{u_1^2} - \frac{h(2 + Y_2 + Y_3)}{2 \cdot 4^k u_1^2 u_2^2}\right) \\ &\quad + O(x^{-1949}) \\ &= B_1(x, \mathbf{U}) + B_2(x, \mathbf{U}) + O(x^{-1949}), \end{aligned}$$

where $B_1(x, \mathbf{U})$ and $B_2(x, \mathbf{U})$ are used to denote contributions from $h = 0$ and $h \neq 0$, respectively.

First,

$$B_1(x, \mathbf{U}) = (1 - \delta) \sum_{\substack{u_1 \sim U_1, u_2 \sim U_2 \\ u_1 \equiv \xi_1, u_2 \equiv \xi_2 \pmod{4^k} \\ (2, u_1 u_2) = (u_1, u_2) = 1}} \frac{Y_3 - Y_2}{4^k u_1^2 u_2^2},$$

which is $\asymp \sqrt{x}$. It is also desirable to show that

$$(3.4) \quad B_2(x, \mathbf{U}) \ll x^{\frac{1}{2} - \varepsilon_0}$$

for some $\varepsilon_0 > 0$. By standard tools from analysis (see [Fo16] for details), it suffices to prove that

$$(3.5) \quad \sum_{\substack{h \leq H \\ h \equiv h_0 \pmod{4^k}}} \sum_{\substack{u_1 \in (U_1, U_1^*] \\ u_1 \equiv \xi_1, u_2 \equiv \xi_2 \pmod{4^k} \\ (2, u_1 u_2) = (u_1, u_2) = 1}} \sum_{u_2 \in (U_2, U_2^*]} e\left(2h \frac{\overline{4^k u_2^2}}{u_1^2}\right) \ll U_1 U_2 x^{-\varepsilon_0},$$

where $U_j < U_j^* \leq 2U_j$, $j = 1, 2$. After transforming the u_2 -sum to a complete sum $T(\cdot, u_1^2)$, where

$$T(h, q) = \sum_{x \pmod{q}}^* e\left(\frac{hx^2}{q}\right) = \sum_{x \pmod{q}}^* e\left(\frac{hx^2}{q}\right),$$

Fouvry [Fo16] evaluated $T(\cdot, u_1^2)$ in terms of classical Gauss sums and Jacobi symbols. He then arrived at a bilinear form involving Jacobi symbols, for which a celebrated estimate due to Heath-Brown [HB95] is applied. Amongst some other delicate arguments, Fouvry was able to prove (3.4) under the conditions

$$(3.6) \quad U_1 \leq x^{\frac{1}{4}-5\varepsilon_0}, \quad U_2 \leq U_1 x^{\frac{1}{2}-5\varepsilon_0},$$

in which case he obtained the lower bound

$$(3.7) \quad B(x, \mathbf{U}) \geq (1 - \delta) \sum_{\substack{u_1 \sim U_1, u_2 \sim U_2 \\ u_1 \equiv \xi_1, u_2 \equiv \xi_2 \pmod{4^k} \\ (2, u_1 u_2) = (u_1, u_2) = 1}} \frac{Y_3 - Y_2}{4^k u_1^2 u_2^2} + O(x^{\frac{1}{2}-\varepsilon_0}).$$

To obtain a better lower bound for $B(x, \alpha)$ and thus for $S(x, \alpha)$, it is natural to expect that (3.4) and (3.7) can hold in larger ranges of U_1, U_2 . However, it seems rather difficult when U_1 is quite close to U_2 since the u_2 -sum is too short in the sense of the Pólya-Vinogradov barrier. In fact, Bourgain [Bo14] managed to control the LHS in (3.5), but with a saving of a small power of $\log x$ rather than that of x . This allows him to improve upon Fouvry when α is rather close to $\frac{1}{2}$ in Theorem B.

In our subsequent argument, we will specialize u_1 with special structures in the original sum (3.3) before Poisson summation. More precisely, we will consider those u_1 with only small prime factors, so that u_1 has good factorizations, which enable us to control the exponential sums in (3.5) even though U_1 is quite close to U_2 .

3.2. Lower bound of $B(x, \mathbf{U})$: smooth approach. A positive integer n is said to be y -smooth (or friable) if all prime factors of n do not exceed y . Let $\theta \in (0, \frac{1}{2})$ be a fixed number. If n is n^θ -smooth, the inclusion-exclusion principle yields the existence of the divisor $d \mid n$ such that $n^{\theta_0} \leq d \leq n^{\theta_0+\theta}$ for any $\theta_0 \in [0, 1 - \theta]$.

We now restrict these u_1 in the RHS of (3.3) to U_1^θ -smooth numbers and put

$$(3.8) \quad B^*(x, \mathbf{U}) = \sum_{\substack{u_1 \sim U_1, u_2 \geq 1 \\ u_1 \equiv \xi_1, u_2 \equiv \xi_2 \pmod{4^k} \\ (2, u_1 u_2) = (u_1, u_2) = 1 \\ u_1 \text{ is } U_1^\theta\text{-smooth}}} g_1\left(\frac{u_2}{U_2}\right) \sum_{\substack{t \in \mathbf{Z} \\ t \equiv t_0 \pmod{4^k u_1^2 u_2^2}}} g\left(\frac{t - \frac{Y_2 + Y_3}{2}}{Y_3 - Y_2}\right),$$

where $g_1(y) = g(y - \frac{3}{2})$ with g given as in Lemma 3.1. Following the similar arguments of Fouvry, we may derive that

$$B^*(x, \mathbf{U}) \geq B_1^*(x, \mathbf{U}) + B_2^*(x, \mathbf{U}) + O(x^{-1949}),$$

where

$$B_1^*(x, \mathbf{U}) = (1 - \delta) \sum_{\substack{u_1 \sim U_1, u_2 \geq 1 \\ u_1 \equiv \xi_1, u_2 \equiv \xi_2 \pmod{4^k} \\ (2, u_1 u_2) = (u_1, u_2) = 1 \\ u_1 \text{ is } U_1^\theta\text{-smooth}}} g_1\left(\frac{u_2}{U_2}\right) \frac{Y_3 - Y_2}{4^k u_1^2 u_2^2}$$

$$\geq (1-\delta)^2 \sum_{\substack{u_1 \sim U_1, u_2 \sim U_2 \\ u_1 \equiv \xi_1, u_2 \equiv \xi_2 \pmod{4^k} \\ (2, u_1 u_2) = (u_1, u_2) = 1 \\ u_1 \text{ is } U_1^\theta\text{-smooth}}} \frac{Y_3 - Y_2}{4^k u_1^2 u_2^2},$$

and we expect to show that

$$(3.9) \quad B_2^*(x, \mathbf{U}) \ll x^{\frac{1}{2}-\varepsilon_0}$$

for some $\varepsilon_0 > 0$, for which it suffices to prove, for all $U_1^* \in (U_1, 2U_1]$, that

$$(3.10) \quad \sum_{\substack{h \leq H \\ h \equiv h_0 \pmod{4^k}}} \sum_{\substack{u_1 \in (U_1, U_1^*] \\ u_1 \equiv \xi_1, u_2 \equiv \xi_2 \pmod{4^k} \\ (2, u_1 u_2) = (u_1, u_2) = 1 \\ u_1 \text{ is } U_1^\theta\text{-smooth}}} \sum_{u_2 \geq 1} g_1\left(\frac{u_2}{U_2}\right) e\left(2h \frac{\overline{4^k u_2^2}}{u_1^2}\right) \ll U_1 U_2 x^{-\varepsilon_0}.$$

We will prove

Theorem 3.1. *There exists some $\varepsilon_0 > 0$ such that (3.10) holds, provided that*

$$(3.11) \quad U_1^{\frac{8+2\theta}{3}} U_2 \leq x^{1-2\varepsilon_0}, \quad U_2 \leq U_1 x^{\frac{1}{2}-5\varepsilon_0}.$$

Put $U_1 = x^{\gamma_1}$ and $U_2 = x^{\gamma_2}$. In addition to the restrictions $\gamma_2 < \gamma_1 + \frac{1}{2}$, $\frac{1}{2} < \gamma_1 + \gamma_2 < \alpha$, (3.6) requires $\gamma_1 < \frac{1}{4}$, and we require $\gamma_2 < 1 - \frac{26}{9}\gamma_1$ in the particular case $\theta = \frac{1}{36}$. In the following figure, the blue area shows what we can gain more than the previous approach (We are gaining relatively more as α becomes closer to $\frac{1}{2}$).

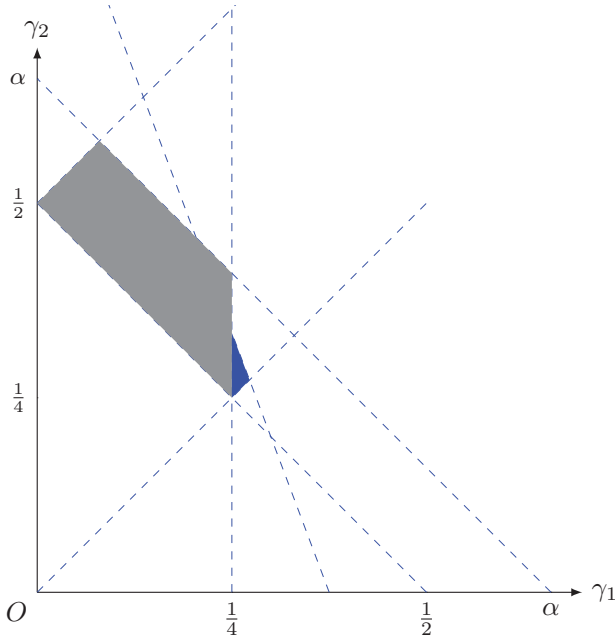


FIGURE 1. Admissible values of (γ_1, γ_2)

The proof of Theorem 3.1 will be given in the next section. To see the advantage of our approach, one may consider the particular case $U_1 = U_2$, and our first restriction will reduce to $U_1 \leq x^{\frac{3}{11+2\theta}-\varepsilon_0}$; however, the stronger restriction $U_1 \leq x^{\frac{1}{4}-5\varepsilon_0}$ in (3.6) is required.

Therefore, we may obtain the lower bound

$$(3.12) \quad B(x, \mathbf{U}) \geq (1 - \delta)^2 \sum_{\substack{u_1 \sim U_1, u_2 \sim U_2 \\ u_1 \equiv \xi_1, u_2 \equiv \xi_2 \pmod{4^k} \\ (2, u_1 u_2) = (u_1, u_2) = 1 \\ u_1 \text{ is } U_1^\theta\text{-smooth}}} \frac{Y_3 - Y_2}{4^k u_1^2 u_2^2} + O(x^{\frac{1}{2}-\varepsilon_0})$$

subject to the restrictions in (3.11).

3.3. A weakened form of Theorem 1.1. Up to now, we have two lower bounds for $B(x, \mathbf{U})$, i.e., (3.7) and (3.12), subject to subject to the restrictions in (3.6) and (3.11), respectively. In what follows, we will take into account all admissible tuples $(k, \xi, \xi_1, \xi_2, U_1, U_2)$, for which we appeal to (3.7) if (3.6) is satisfied, and appeal to (3.12) if (3.6) is not satisfied but (3.11) is valid. To this end, we define two sets

$$\mathcal{U}_1 := \{(U_1, U_2) : U_1 < U_2, \frac{\sqrt{x}}{2} < 2^k U_1 U_2 \leq \frac{x^\alpha}{16}, U_1 \leq x^{\frac{1}{4}-\eta}, U_2 \leq U_1 x^{\frac{1}{2}-\eta}\},$$

and

$$\mathcal{U}_2 := \{(U_1, U_2) : U_1 < U_2, \frac{\sqrt{x}}{2} < 2^k U_1 U_2 \leq \frac{x^\alpha}{16}, U_1^{\frac{8+2\theta}{3}} U_2 \leq x^{1-\eta}, U_2 \leq U_1 x^{\frac{1}{2}-\eta}\} \setminus \mathcal{U}_1,$$

where η is a sufficiently small positive number.

First, we may derive a lower bound for $B^<(x, \alpha; \xi, k)$ by inserting the inequality (3.7) or (3.12) to (3.1). A similar lower bound also holds for $B^>(x, \alpha; \xi, k)$ by symmetry. Therefore, we have

$$\begin{aligned} B(x, \alpha) &\geq 2(1 - \delta) \sum_{k=0}^{k_0} \sum_{\xi \in \mathcal{R}(2^k)} \sum_{\mathbf{U} \in \mathcal{U}_1} \sum_{\substack{\xi_1, \xi_2 \pmod{4^k} \\ 2 \nmid \xi_1 \xi_2}} \sum_{\substack{u_1 \sim U_1, u_2 \sim U_2 \\ u_1 \equiv \xi_1, u_2 \equiv \xi_2 \pmod{4^k} \\ (2, u_1 u_2) = (u_1, u_2) = 1}} \frac{Y_3 - Y_2}{4^k u_1^2 u_2^2} \\ &\quad + 2(1 - 2\delta) \sum_{k=0}^{k_0} \sum_{\xi \in \mathcal{R}(2^k)} \sum_{\mathbf{U} \in \mathcal{U}_2} \sum_{\substack{\xi_1, \xi_2 \pmod{4^k} \\ 2 \nmid \xi_1 \xi_2}} \sum_{\substack{u_1 \sim U_1, u_2 \sim U_2 \\ u_1 \equiv \xi_1, u_2 \equiv \xi_2 \pmod{4^k} \\ (2, u_1 u_2) = (u_1, u_2) = 1 \\ u_1 \text{ is } U_1^\theta\text{-smooth}}} \frac{Y_3 - Y_2}{4^k u_1^2 u_2^2} + O(x^{\frac{1}{2}}), \end{aligned}$$

where U_1, U_2 are also restricted to be powers of 2. Recall that

$$Y_3 - Y_2 = Y_3(2^k u_1 u_2, \alpha) - Y_2(2^k u_1 u_2) = 2^k u_1 u_2 \sqrt{x} + O((2^k u_1 u_2)^{1+\frac{1}{2\alpha}}).$$

We then obtain the lower bound

$$\begin{aligned} B(x, \alpha) &\geq 2(1 - 2\delta) \left(\sum_{k=0}^{k_0} \frac{\gamma(2^k)}{2^k} \right) \left\{ \sum_{\substack{(u_1, u_2) \in \mathcal{Y}_1 \\ (2, u_1 u_2) = (u_1, u_2) = 1}} \frac{1}{u_1 u_2} + \sum_{\substack{(u_1, u_2) \in \mathcal{Y}_2 \\ (2, u_1 u_2) = (u_1, u_2) = 1 \\ u_1 \text{ is } U_1^\theta\text{-smooth}}} \frac{1}{u_1 u_2} \right\} \\ &\quad + O(x^{\frac{1}{2}} \log x), \end{aligned}$$

where

$$\mathcal{V}_1 := \{(u_1, u_2) : u_1 < u_2, \sqrt{x} < u_1 u_2 \leq x^\alpha, u_1 \leq x^{\frac{1}{4}-\eta}, u_2 \leq u_1 x^{\frac{1}{2}-\eta}\},$$

and

$$\mathcal{V}_2 := \{(u_1, u_2) : u_1 < u_2, \sqrt{x} < u_1 u_2 \leq x^\alpha, u_1^{\frac{s+2\theta}{3}} u_2 \leq x^{1-\eta}, u_2 \leq u_1 x^{\frac{1}{2}-\eta}\} \setminus \mathcal{V}_1,$$

Taking $k_0 = k_0(\delta)$ very large, $\eta = \eta(\delta)$ very small, and letting δ tend to zero, we conclude from Lemma A.1 that

$$(3.13) \quad B(x, \alpha) \geq \frac{1}{\pi^2}((2\alpha - 1)(3 - 2\alpha) - o(1))\sqrt{x} \log^2 x + B'(x, \alpha)$$

uniformly for $\alpha \in [\frac{1}{2}, 1]$, where

$$B'(x, \alpha) = 4\sqrt{x} \sum_{\substack{x^{\frac{1}{4}} < u_1 \leq x^{\min\{\frac{\alpha}{2}, \frac{3}{11+2\theta}\}} \\ u_1 \text{ is } U_1^\theta\text{-smooth} \\ 2 \nmid u_1}} \frac{\varphi(u_1)}{u_1^2} \log(\sqrt{x} u_1^{-\frac{5+2\theta}{3}}).$$

Note that $B'(x, \alpha)$ is what we have gained more than Fouvry [Fo16]. From Lemma A.3, we arrive at

$$B'(x, \alpha) = \frac{1}{\pi^2}(F(\alpha, \theta) + o(1))\sqrt{x} \log^2 x$$

with

$$F(\alpha, \theta) = \rho\left(\frac{1}{\theta}\right) \left\{ 4 \min \left\{ \alpha, \frac{6}{11+2\theta} \right\} - \frac{(10+4\theta)}{3} \min \left\{ \alpha, \frac{6}{11+2\theta} \right\}^2 - \frac{(7-2\theta)}{6} \right\}.$$

One may check that

$$F(\alpha, \theta) = \frac{1}{6} \rho\left(\frac{1}{\theta}\right) F_\theta(\alpha)$$

as given in Theorem 1.1. Combining this asymptotic evaluation for $B'(x, \alpha)$ with (3.13), we may conclude a lower bound for $B(x, \alpha)$, from which and (2.8) and (2.7), we get

$$(3.14) \quad S(x, \alpha) \geq \frac{1}{\pi^2} \left\{ 1 + (2\alpha - 1)(3 - 2\alpha) + \frac{1}{6} \rho\left(\frac{1}{\theta}\right) F_\theta(\alpha) - o(1) \right\} \sqrt{x} \log^2 x$$

uniformly for $\alpha \in [\frac{1}{2}, 1]$.

3.4. Concluding Theorem 1.1. To pass from a lower bound of $S(x, \alpha)$ to that of $S^f(x, \alpha)$, it is natural to invoke the identity (2.13) and Theorem A. In fact, one can do a bit better following the arguments of Fouvry [Fo16] and show that the above lower bound (3.14) also hold for $S^f(x, \alpha)$. This will depend on a more elaborate study of the contribution of the non-fundamental solutions. In other words, we would like to show that the non-fundamental solutions create negligible contributions to $A(x, \alpha)$ and $B(x, \alpha)$.

The following lemma is borrowed directly from [Fo16, Lemma 9.1].

Lemma 3.2. *Uniformly for $\alpha \in [\frac{1}{2}, 1]$ and $x \geq 2$, one has*

$$A(x, \alpha) - A^f(x, \alpha) \ll \sqrt{x} \log x.$$

To deal with the contribution of the non-fundamental solutions to $B(x, \alpha)$, we also follow Fouvry. The above arguments which lead to (3.14) are essentially counting the number $\mathcal{N}(x, \alpha; \varepsilon, k_0)$ of 5-tuples of integers (k, t, u_1, u_2, D) satisfying

$$\begin{aligned} 2 \nmid u_1 u_2, \quad (u_1, u_2) &= 1, \quad 0 \leq k \leq k_0, \quad D \leq x, \\ X_{\frac{1}{2}} &\leq 2^k u_1 u_2 \leq X_\alpha, \quad t + 2^k u_1 u_2 \sqrt{D} \leq D^{\frac{1}{2} + \alpha}, \\ t^2 &\equiv 1 \pmod{4^k}, \quad t \equiv 1 \pmod{u_1^2}, \quad t \equiv -1 \pmod{u_2^2}, \end{aligned}$$

as well as one of the following restrictions:

- $u_1 \leq u_2 \leq u_1 x^{\frac{1}{2} - \eta}, \quad u_1 \leq x^{\frac{1}{4} - \eta};$
- $u_2 \leq u_1 \leq u_2 x^{\frac{1}{2} - \eta}, \quad u_2 \leq x^{\frac{1}{4} - \eta};$
- $u_1 \leq u_2 \leq u_1 x^{\frac{1}{2} - \eta}, \quad u_1^{\frac{s+2\theta}{3}} u_2 \leq x^{1-\eta};$
- $u_2 \leq u_1 \leq u_2 x^{\frac{1}{2} - \eta}, \quad u_1 u_2^{\frac{s+2\theta}{3}} \leq x^{1-\eta}.$

By introducing the extra constraint $t + 2^k u_1 u_2 \sqrt{D} = \varepsilon_D$, we may also define $\mathcal{N}^f(x, \alpha; \varepsilon, k_0)$. In fact, the above arguments yield

$$(3.15) \quad B(x, \alpha) \geq \mathcal{N}(x, \alpha; \varepsilon, k_0), \quad B^f(x, \alpha) \geq \mathcal{N}^f(x, \alpha; \varepsilon, k_0),$$

which are true for every positive $\varepsilon > 0$ and for every $k_0 \geq 0$. More precisely, we have proved for every δ , $0 < \varepsilon < \varepsilon_0(\delta)$, $k_0 > k_0(\delta)$ and $x > x_0(\delta)$ that

$$(3.16) \quad \mathcal{N}(x, \alpha; \varepsilon, k_0) \geq \frac{1}{\pi^2} \left\{ (2\alpha - 1)(3 - 2\alpha) + \frac{1}{6} \rho\left(\frac{1}{\theta}\right) F_\theta(\alpha) - \delta \right\} \sqrt{x} \log^2 x$$

with $\alpha \in [\frac{1}{2}, 1]$.

Following the approach of Fouvry [Fol16], we can state without proof that

Lemma 3.3. *For every $k_0 \geq 0$ and every $\varepsilon > 0$, one has*

$$\mathcal{N}(x, \alpha; \varepsilon, k_0) - \mathcal{N}^f(x, \alpha; \varepsilon, k_0) \ll_{\varepsilon, k_0} \sqrt{x} \log x$$

uniformly for $\alpha \in [\frac{1}{2}, 1]$ and $x \geq 2$.

We are now ready to complete the proof of Theorem 1.1. In view of (3.15), we may write

$$\begin{aligned} S^f(x, \alpha) &= A^f(x, \alpha) + B^f(x, \alpha) \\ &\geq A^f(x, \alpha) + \mathcal{N}^f(x, \alpha; \varepsilon, k_0) \\ &= A(x, \alpha) + \mathcal{N}(x, \alpha; \varepsilon, k_0) + \{A^f(x, \alpha) - A(x, \alpha)\} \\ &\quad + \{\mathcal{N}^f(x, \alpha; \varepsilon, k_0) - \mathcal{N}(x, \alpha; \varepsilon, k_0)\}, \end{aligned}$$

which are true for every $\varepsilon > 0$ and $k_0 \geq 0$. From Lemmas 3.2 and 3.3, we obtain

$$S^f(x, \alpha) \geq A(x, \alpha) + \mathcal{N}(x, \alpha; \varepsilon, k_0) + O(\sqrt{x} \log x).$$

By (2.8), (3.16), and by choosing $k_0 = k_0(\delta)$ sufficiently large, $\eta = \eta(\delta)$ sufficiently small, and letting δ tend to zero, we find the lower bound (3.14) holds definitely for $S^f(x, \alpha)$. This establishes (1.4).

The lower bounds for $S(x, \alpha)$ in Theorem 1.1 can be deduced from (1.2) by adding the contribution of the non-fundamental solutions, as it is shown by (2.13).

4. ESTIMATE FOR TRIPLE EXPONENTIAL SUMS

For the economy of the presentation in proving Theorem 3.1, we only focus on the case $k = 0$. We thus define

$$(4.1) \quad \mathfrak{S}(U_1, U_2, H) := \sum_{h \leq H} \sum_{\substack{u_1 \in (U_1, U_1^*] \\ (2, u_1 u_2) = (u_1, u_2) = 1 \\ u_1 \text{ is } U_1^\theta\text{-smooth}}} \sum_{u_2 \geq 1} g_1\left(\frac{u_2}{U_2}\right) e\left(\frac{h \overline{u_2^2}}{u_1^2}\right)$$

and we would like to show that

$$(4.2) \quad \mathfrak{S}(U_1, U_2, H) \ll U_1 U_2 x^{-\varepsilon_0}$$

for some $\varepsilon_0 > 0$ while U_1, U_2 fall into the ranges in (3.11).

By Poisson summation, the u_2 -sum becomes

$$\frac{U_2}{2u_1^2} \sum_{r \in \mathbf{Z}} (-1)^r \widehat{g}_1\left(\frac{rU_2}{u_1^2}\right) K(r, h; u_1^2),$$

where K is an analogue of Kloosterman sums:

$$K(m, n; q) = \sum_{x \pmod{q}}^* e\left(\frac{mx + n\overline{x}^2}{q}\right).$$

Note that $K(0, h; q) = T(h, q)$.

According to $r = 0$ and $r \neq 0$, we split $\mathfrak{S}(U_1, U_2, H)$ by

$$\mathfrak{S}(U_1, U_2, H) = \mathfrak{S}_1(U_1, U_2, H) + \mathfrak{S}_2(U_1, U_2, H),$$

where

$$\mathfrak{S}_1(U_1, U_2, H) = \frac{U_2(1 - \delta)}{2} \sum_{h \leq H} \sum_{\substack{u_1 \in (U_1, U_1^*] \\ 2 \nmid u_1 \text{ is } U_1^\theta\text{-smooth}}} \frac{T(h, u_1^2)}{u_1^2}$$

and

$$\mathfrak{S}_2(U_1, U_2, H) = \frac{U_2}{2} \sum_{h \leq H} \sum_{\substack{u_1 \in (U_1, U_1^*] \\ 2 \nmid u_1 \text{ is } U_1^\theta\text{-smooth}}} \frac{1}{u_1^2} \sum_{0 \neq r \in \mathbf{Z}} (-1)^r \widehat{g}_1\left(\frac{rU_2}{u_1^2}\right) K(r, h; u_1^2).$$

Following the approach of Fouvry, one may express $T(h, u_1^2)$ in terms of Jacobi symbols (see [Fo16, Lemma 6.2]) and then appeal to the bilinear estimate of Heath-Brown [HB95], getting

$$(4.3) \quad \mathfrak{S}_1(U_1, U_2, H) \ll (U_1 U_2 H)^\varepsilon (H U_1 U_2^{-1} + H^{\frac{1}{2}} U_1),$$

which produces the second restriction in (3.6). Moreover, Fouvry proved that

$$\mathfrak{S}_2(U_1, U_2, H) \ll (U_1 U_2 H)^\varepsilon H U_2^2,$$

which produces the first restriction in (3.6).

Our task will be proving a stronger estimate for $\mathfrak{S}_2(U_1, U_2, H)$ by virtue of the special structure of u_1 . More precisely, we shall prove that

$$\mathfrak{S}_2(U_1, U_2, H) \ll (U_1 U_2 H)^\varepsilon H U_1 U_2^{\frac{1}{2}} (Q + U_1^{\frac{1}{2}} Q^{-\frac{1}{2}}),$$

where Q will be chosen at our demand. This would at least require the following inequality as proved by Fouvry [F016]. In fact, Fouvry only considered q which is a perfect square, and his argument also applies to other q .

Lemma 4.1. *Let q be an odd positive integer. Then we have*

$$|K(m, n; q)| \leq 3^{\omega(q)}(m, n, q)\sqrt{q}.$$

As an extension to $K(m, n; q)$, we define another exponential sum

$$\mathcal{B}(m, n, \ell, u; q) = \sum_{a \pmod{q}}^* e\left(\frac{m\overline{a^2} + n\overline{(a+u)^2} + \ell a}{q}\right).$$

We will need the following inequality.

Lemma 4.2. *For each odd $q \geq 1$, we have*

$$|\mathcal{B}(m, n, \ell, u; q)| \leq 18^{\omega(q)}\sqrt{q}(mu, \ell, q).$$

To prove Lemma 4.2, we need the following auxiliary results. The first one is a consequence of the Chinese remainder theorem, which shows it suffices to consider the case of prime power moduli in Lemma 4.2.

Lemma 4.3. *For $q = q_1 q_2$ with $(q_1, q_2) = 1$, we may write*

$$\mathcal{B}(m, n, \ell, u; q) = \mathcal{B}(m\overline{q_2^3}, n\overline{q_2^3}, \ell, u\overline{q_2}; q_1) \mathcal{B}(m\overline{q_1^3}, n\overline{q_1^3}, \ell, u\overline{q_1}; q_2),$$

where $q_1\overline{q_1} \equiv 1 \pmod{q_2}$ and $q_2\overline{q_2} \equiv 1 \pmod{q_1}$.

The following lemma is taken from [Es61, Lemma 1].

Lemma 4.4. *Let α and β be positive integers such that*

$$\frac{\beta}{2} \leq \alpha < \beta,$$

and s and p such that $p \nmid s$. Then we have the equality

$$\overline{s + p^\alpha t} \equiv \overline{s} - p^\alpha t \overline{s^2} \pmod{p^\beta},$$

where all the inverses are taken modulo p^β .

We are ready to prove Lemma 4.2 by considering the case of prime power moduli.

Case I: Even powers.

$$\begin{aligned} \mathcal{B}(m, n, \ell, u; p^{2\alpha}) &= \sum_{a \pmod{p^{2\alpha}}}^* e\left(\frac{m\overline{a^2} + n\overline{(a+u)^2} + \ell a}{p^{2\alpha}}\right) \\ &= \sum_{a \pmod{p^\alpha}}^* \sum_{b \pmod{p^\alpha}} e\left(\frac{m\overline{(a+bp^\alpha)^2} + n\overline{(a+bp^\alpha+u)^2} + \ell(a+bp^\alpha)}{p^{2\alpha}}\right). \end{aligned}$$

From Lemma 4.4 we may write

$$\begin{aligned} \mathcal{B}(m, n, \ell, u; p^{2\alpha}) &= \sum_{a \pmod{p^\alpha}}^* e\left(\frac{m\overline{a^2} + n\overline{(a+u)^2} + \ell a}{p^{2\alpha}}\right) \\ &\quad \times \sum_{b \pmod{p^\alpha}} e\left(\frac{b(2m\overline{a^3} + 2n\overline{(a+u)^3} - \ell)}{p^\alpha}\right). \end{aligned}$$

The orthogonality of additive characters gives

$$\mathcal{B}(m, n, \ell, u; p^{2\alpha}) = p^\alpha \sum_{\substack{a \pmod{p^\alpha} \\ 2m\overline{a^3} + 2n\overline{(a+u)^3} \equiv \ell \pmod{p^\alpha}}}^* e\left(\frac{\overline{ma^2} + n\overline{(a+u)^2} + \ell a}{p^{2\alpha}}\right).$$

Note that $2m\overline{a^3} + 2n\overline{(a+u)^3} \equiv \ell \pmod{p^\alpha}$ is equivalent to

$$\ell a^6 + 3\ell u a^5 + 3\ell u^2 a^4 + (\ell u^3 - 2m - 2n)a^3 - 6mu a^2 - 6mu^2 a - 2mu^3 \equiv 0 \pmod{p^\alpha}.$$

As a congruence equation in $a \pmod{p}^\alpha$, there are at most $6(\ell, 3mu, p^\alpha)$ solutions. This yields

$$|\mathcal{B}(m, n, \ell, u; p^{2\alpha})| \leq 18p^\alpha(mu, \ell, p^\alpha).$$

Case II: Odd powers. First, we consider the case of prime moduli.

$$\mathcal{B}(m, n, \ell, u; p) = \sum_{a \pmod{p}}^* e\left(\frac{\overline{ma^2} + n\overline{(a+u)^2} + \ell a}{p}\right).$$

For $u = 0$, we have $\mathcal{B}(m, n, \ell, 0; p) = K(\ell, m+n; p)$, from which and Lemma 4.1, we get

$$|\mathcal{B}(m, n, \ell, 0; p)| \leq 3(\ell, m+n, p)\sqrt{p} = 3(mu, \ell, p)\sqrt{p}.$$

For $p \nmid u$, we can write

$$\mathcal{B}(m, n, \ell, u; p) = \sum_{a \pmod{p}}^* e\left(\frac{\phi(a)}{p}\right)$$

with

$$\phi(x) = \frac{\ell x^5 + 2\ell u x^4 + \ell u^2 x^3 + (m+n)x^2 + 2mux + mu^2}{x^2(x+u)^2}.$$

From a general estimate for exponential sums over finite fields due to Weil [We48], we find

$$|\mathcal{B}(m, n, \ell, u; p)| \leq (\deg \phi - 1)\sqrt{p} \leq 8\sqrt{p}.$$

For the case of $q = p^{2\alpha+1}$ with $\alpha \geq 1$, one can follow the arguments in Case I and get

$$|\mathcal{B}(m, n, \ell, u; p^{2\alpha+1})| \leq 18p^\alpha(mu, \ell, p^{\alpha+1}).$$

Combining the above cases, we arrive at the uniform inequality

$$|\mathcal{B}(m, n, \ell, u; q)| \leq 18^{\omega(q)}\sqrt{q}(mu, \ell, q).$$

This completes the proof of Lemma 4.2.

We now start to prove Theorem 3.1. Due to the decay of \widehat{g}_1 , we may truncate the r -sum in $\mathfrak{S}_2(U_1, U_2, H)$ by $R = U_1^{2+\varepsilon}U_2^{-1}$, so that

$$(4.4) \quad \mathfrak{S}_2(U_1, U_2, H) = \frac{U_2}{2} \sum_{h \leq H} \sum_{\substack{u_1 \in (U_1, U_1^*] \\ 2 \nmid u_1 \text{ is } U_1^\ell\text{-smooth}}} \frac{1}{u_1^2} \sum_{1 \leq |r| \leq R} (-1)^r \widehat{g}_1\left(\frac{rU_2}{u_1^2}\right) K(r, h; u_1^2) \\ + O((U_1 U_2 H)^{-2017}).$$

Our project will be controlling the cancellations while summing over r with $1 \leq |r| \leq R$. By partial summation, it suffices to consider

$$\Sigma(R, u_1^2; h) := \sum_{r \leq R} K(r, h; u_1^2).$$

For each $u_0 \mid u_1$, we define d, q_1, q_2 by

$$d = (u_0, (u_1/u_0)^\infty), \quad q_1 = (u_0/d)^2, \quad q_2 = u_1^2/q_1.$$

It follows that $u_1^2 = q_1 q_2$ and $(q_1, q_2) = 1$. By virtue of the q -analogue of van der Corput method, we will prove

Lemma 4.5. *With the above notation, we have*

$$\Sigma(R, u_1^2; h) \ll u_1^\varepsilon R^{\frac{1}{2}} \{u_0 u_1 + u_0 R^{\frac{1}{2}} (u_0, u_1/u_0) + (u_0, (u_1/u_0)^\infty)^{\frac{1}{2}} (h, u_0^2)^{\frac{1}{2}} u_1^{\frac{3}{2}}/u_0^{\frac{1}{2}}\}.$$

Proof. Lemma 4.5 is a consequence of Lemma 4.1 if $R \leq u_0^2$. We now assume $R > u_0^2$. Denote by I_R the characteristic function of the interval $[1, R]$. Thus,

$$\Sigma(R, u_1^2; h) = \sum_{r \in \mathbf{Z}} I_R(r) K(r, h; u_1^2) = \sum_{r \in \mathbf{Z}} I_R(r + u_0^2 \ell) K(r + u_0^2 \ell, h; q)$$

for any $\ell \in \mathbf{Z}$. From the Chinese remainder theorem, we may write

$$\begin{aligned} K(r + u_0^2 \ell, h; u_1^2) &= K(r + u_0^2 \ell, h \bar{q}_2^3; q_1) K(r + u_0^2 \ell, h \bar{q}_1^3; q_2) \\ &= K(r, h \bar{q}_2^3; q_1) K(r + u_0^2 \ell, h \bar{q}_1^3; q_2), \end{aligned}$$

where we have used the fact that $q_1 \mid u_0^2$.

For $L = [R/u_0^2]$, we sum over ℓ , getting

$$\begin{aligned} \Sigma(R, u_1^2; h) &= \frac{1}{L} \sum_{\ell \leq L} \sum_{r \in \mathbf{Z}} I_R(r + u_0^2 \ell) K(r + u_0^2 \ell, h; q) \\ &\leq \frac{1}{L} \sum_{r \in \mathbf{Z}} |K(r, h \bar{q}_2^3; q_1)| \left| \sum_{\ell \leq L} I_R(r + u_0^2 \ell) K(r + u_0^2 \ell, h \bar{q}_1^3; q_2) \right|. \end{aligned}$$

From Lemma 4.1 we find

$$\Sigma(R, u_1^2; h) \ll \frac{q^\varepsilon \sqrt{q_1}}{L} \sum_{r \in \mathbf{Z}} |K(r, h, q_1)| \left| \sum_{\ell \leq L} I_R(r + u_0^2 \ell) K(r + u_0^2 \ell, h \bar{q}_1^3; q_2) \right|.$$

In view of the support of I_R , the sum over r is in fact restricted to $[-R, R]$. By Cauchy inequality, we derive that

$$\Sigma(R, u_1^2; h)^2 \ll \frac{q^\varepsilon q_1 R}{L^2} \sum_{r \in \mathbf{Z}} \left| \sum_{\ell \leq L} I_R(r + u_0^2 \ell) K(r + u_0^2 \ell, h \bar{q}_1^3; q_2) \right|^2.$$

Squaring out and switching summations, we get

$$(4.5) \quad \Sigma(R, u_1^2; h)^2 \ll \frac{q^\varepsilon q_1 R}{L} \sum_{0 \leq |\ell| \leq L} \left| \sum_{r \in I_\ell} K(r, h \bar{q}_1^3; q_2) \overline{K(r + u_0^2 \ell, h \bar{q}_1^3; q_2)} \right|,$$

where I_ℓ is an interval, depending on ℓ , of length at most R . For $\ell = 0$, we appeal to Lemma 4.1 to estimate the r -sum trivially. For $1 \leq |\ell| \leq L$, reasonable cancellations in the r -sum are expected. In fact, by completion, we have

$$\sum_{r \in I_\ell} [\cdots] = \sum_{y \pmod{q_2}} I(y, q_2) \widehat{K}(y, q_2; h, q_1, u_0, \ell),$$

where

$$I(y, q_2) = \sum_{r \in I_\ell} e\left(\frac{-yr}{q_2}\right),$$

and

$$\widehat{K}(y, q_2; h, q_1, u_0, \ell) = \frac{1}{q_2} \sum_{x \pmod{q_2}} K(x, h\bar{q}_1^3; q_2) \overline{K(x + u_0^2\ell, h\bar{q}_1^3; q_2)} e\left(\frac{yx}{q_2}\right).$$

On one hand,

$$I(y, q_2) \ll \min\left\{R, \left\|\frac{y}{q_2}\right\|^{-1}\right\}.$$

Opening each K by definition, the orthogonality of additive characters gives

$$\begin{aligned} \widehat{K}(y, q_2; h, q_1, u_0, \ell) &= e\left(\frac{-u_0^2\ell y}{q_2}\right) \sum_{a \pmod{q_2}}^* e\left(\frac{h\bar{q}_1^3(\bar{a}^2 - \overline{(a+y)^2}) - u_0^2\ell a}{q_2}\right) \\ &= e\left(\frac{-u_0^2\ell y}{q_2}\right) \mathcal{B}(h\bar{q}_1^3, -h\bar{q}_1^3, -u_0^2\ell, y; q_2). \end{aligned}$$

For $y = 0$, we have

$$\mathcal{B}(h\bar{q}_1^3, -h\bar{q}_1^3, -u_0^2\ell, 0; q_2) = \sum_{a \pmod{q_2}}^* e\left(\frac{u_0^2\ell a}{q_2}\right) \ll (u_0^2\ell, q_2).$$

For $y = 0$, we may appeal to Lemma 4.3, getting

$$\begin{aligned} \sum_{r \in I_\ell} [\cdots] &\ll R(u_0^2\ell, q_2) + q_2^{\frac{1}{2}+\varepsilon} \sum_{1 \leq |y| \leq q_2/2} \min\left\{R, \frac{q_2}{y}\right\} (hy, u_0^2\ell, q_2) \\ &\ll R(u_0^2\ell, q_2) + q_2^{\frac{3}{2}+\varepsilon} (h, u_0^2\ell, q_2), \end{aligned}$$

from which and (4.5) we get

$$\begin{aligned} \Sigma(R, u_1^2; h)^2 &\ll \frac{u_1^\varepsilon q_1 R}{L} \{Rq_2(h, q_2) + LR(u_0^2, q_2) + Lq_2^{\frac{3}{2}}(h, u_0^2, q_2)\} \\ &\ll u_1^\varepsilon R \{(u_0 u_1)^2 + q_1 R(u_0^2, q_2) + u_1^2 q_2^{\frac{1}{2}}(h, u_0^2, q_2)\}. \end{aligned}$$

Note that $q_2 = (u_1 d / u_0)^2 = \{u_1(u_0, (u_1/u_0)^\infty / u_0)\}^2$. We find

$$\Sigma(R, u_1^2; h)^2 \ll u_1^\varepsilon R \{(u_0 u_1)^2 + u_0^2 R(u_0, u_1/u_0)^2 + (u_0, (u_1/u_0)^\infty)(h, u_0^2) u_1^3 / u_0\},$$

which gives Lemma 4.5 immediately. \square

To conclude from Lemma 4.5 an estimate for $\mathfrak{S}_2(U_1, U_2, H)$, we shall use the following classical inequality as a consequence of Rankin's method, which is already stated in [Fo16, Lemma 7.2].

Lemma 4.6. *For any $\varepsilon > 0$, one has*

$$\sum_{mn \leq N} (m, n^\infty)^{\frac{1}{2}} \ll N^{1+\varepsilon}.$$

Proof. First, we have

$$\begin{aligned} \sum_{mn \leq N} (m, n^\infty)^{\frac{1}{2}} &\ll \sum_{n \leq N} \sum_{d|n^\infty} d^{\frac{1}{2}} \sum_{\substack{m \leq N/n \\ d|m}} 1 \\ &\leq N \sum_{n \leq N} \frac{1}{n} \sum_{\substack{d \leq N \\ d|n^\infty}} d^{-\frac{1}{2}}. \end{aligned}$$

As a consequence of [Fo16, Lemma 7.2], the inner sum over d is at most $O((nN)^\varepsilon)$, from which the lemma follows immediately. \square

In view of Lemma 4.5, we may derive from (4.4) that

$$\mathfrak{S}_2(U_1, U_2, H) \ll U_1^\varepsilon H Q U_2 (R^{\frac{1}{2}} + R U_1^{-1}) + \frac{U_1^\varepsilon H U_2 R^{\frac{1}{2}}}{(Q U_1)^{\frac{1}{2}}} \sum_{u_1 \sim U_1} \sum_{ab=u_1} (a, b^\infty)^{\frac{1}{2}}.$$

where $U_1^{\theta_0} \leq Q \leq U_1^{\theta_0+\theta}$. In view of the choice $R = U_1^{2+\varepsilon} U_2^{-1}$, Lemma 4.6 yields

$$\mathfrak{S}_2(U_1, U_2, H) \ll U_1^\varepsilon H U_1 U_2^{\frac{1}{2}} \{Q + U_1^{\frac{1}{2}} Q^{-\frac{1}{2}}\},$$

from which and (4.3) we conclude that

$$\mathfrak{S}(U_1, U_2, H) \ll (U_1 U_2 H)^\varepsilon \{H U_1 U_2^{-1} + H^{\frac{1}{2}} U_1 + H U_1 U_2^{\frac{1}{2}} (Q + U_1^{\frac{1}{2}} Q^{-\frac{1}{2}})\}.$$

Choosing $\theta_0 = \frac{1-2\theta}{3}$, we then have $U_1^{\frac{1-2\theta}{3}} \leq Q \leq U_1^{\frac{1+\theta}{3}}$. Recalling the choice of H , we then arrive at the expected estimate (4.2), provided that (3.11) holds.

APPENDIX A. MEAN VALUES OF ARITHMETIC FUNCTIONS

A.1. Some basic asymptotics. The first part of the appendix is devoted to state several basic asymptotics.

Lemma A.1. *As $N \rightarrow +\infty$, we have*

$$\sum_{0 \leq n \leq N} \frac{\gamma(2^n)}{2^n} = 4 + O(2^{-N}),$$

$$\sum_{\substack{n \leq N \\ (n, q)=1}} \frac{1}{n} = \frac{\varphi(q)}{q} \log N + O\left(1 + \sum_{d|q} \frac{\log d}{d}\right),$$

$$\sum_{\substack{n \leq N \\ (n, 2)=1}} \frac{\varphi(n)}{n^2} \sim \frac{4}{\pi^2} \log N,$$

and

$$\sum_{\substack{n \leq N \\ (n, 2)=1}} \frac{\varphi(n) \log n}{n^2} \sim \frac{2}{\pi^2} \log^2 N.$$

Proof. The first one can be derived from the evaluations of $\gamma(2^n)$ as given in (2.10). The other three asymptotics can be found in Fouvry [Fo16, Lemma 8.1, 8.2]. \square

A.2. Smooth numbers. Denote by $\mathcal{S}(x, y)$ the set of y -smooth numbers not exceeding x . Write $\Psi(x, y) = |\mathcal{S}(x, y)|$. We now introduce the Dickman function $\rho(u)$ by

$$(A.1) \quad \begin{cases} \rho(u) = 1, & u \in (0, 1], \\ u\rho'(u) + \rho(u-1) = 0, & u \in (1, +\infty). \end{cases}$$

In the first several intervals, we have

$$\begin{aligned} \rho(u) &= 1 - \log u, & u \in (1, 2], \\ \rho(u) &= 1 - \log u + \int_2^u \frac{\log(t-1)}{t} dt, & u \in (2, 3], \\ \rho(u) &= 1 - \log u + \int_2^u \frac{\log(t-1)}{t} dt - \int_3^u \frac{dt}{t} \int_2^{t-1} \frac{\log(s-1)}{s} ds, & u \in (3, 4]. \end{aligned}$$

The following lemma is classical and shows ρ is the density function of smooth numbers.

Lemma A.2. *Uniformly for $x \geq y \geq 2$, we have*

$$\Psi(x, y) = x\rho\left(\frac{\log x}{\log y}\right) + O\left(\frac{x}{\log y}\right).$$

Proof. See [Te95, P. 367, Theorem 6]. \square

Lemma A.3. *Let $\theta \in (0, 1)$ be fixed. As $N \rightarrow +\infty$, we have*

$$\sum_{\substack{2 \nmid n \leq N \\ n \text{ is } n^\theta\text{-smooth}}} \frac{\varphi(n)}{n^2} \sim \frac{4}{\pi^2} \rho\left(\frac{1}{\theta}\right) \log N,$$

and

$$\sum_{\substack{2 \nmid n \leq N \\ n \text{ is } n^\theta\text{-smooth}}} \frac{\varphi(n) \log n}{n^2} \sim \frac{2}{\pi^2} \rho\left(\frac{1}{\theta}\right) \log^2 N.$$

Proof. One can refer to [TW03], for instance, for some general theorems on the mean values of multiplicative functions over smooth numbers. In particular, one has

$$\sum_{\substack{2 \nmid n \leq N \\ n \text{ is } n^\theta\text{-smooth}}} \frac{\varphi(n)}{n} = \frac{4}{\pi^2} \rho\left(\frac{1}{\theta}\right) N + O\left(\frac{N}{\log N}\right).$$

The lemma then follows from the partial summation. \square

REFERENCES

- [Bo14] J. Bourgain, A remark on solutions of the Pell equation, *Int. Math. Res. Not.* 2015, 2841–2855.
- [Es61] T. Estermann, On Kloosterman’s sum, *Mathematika* **8** (1961), 83–86.
- [Fo16] É. Fouvry, On the size of the fundamental solution of the Pell equation, *J. Reine Angew. Math.* **717** (2016), 1–33.
- [FJ12] E. Fouvry & F. Jouve, Fundamental solutions to Pell equation with prescribed size, *Proc. Steklov Inst. Math.* **276** (2012), 40–50.
- [HB78] D. R. Heath-Brown, Hybrid bounds for Dirichlet L -functions, *Invent. Math.* **47** (1978), 149–170.
- [HB95] D. R. Heath-Brown, A mean value estimate for real character sums, *Acta Arith.* **72** (1995), 235–275.
- [Ho82] C. Hooley, On the Pellian equation and the class number of indefinite binary quadratic forms, *J. Reine Angew. Math.* **353** (1984), 98–131.
- [Te95] G. Tenenbaum, Introduction to Analytic and Probabilistic Number Theory, Cambridge Stud. Adv. Math. 46, Cambridge University Press, Cambridge 1995.
- [TW03] G. Tenenbaum & J. Wu, Moyennes de certaines fonctions multiplicatives sur les entiers friables, *J. Reine Angew. Math.* **564** (2003), 119–166.
- [We48] A. Weil, On some exponential sums, *Proc. Nat. Acad. Sci. U.S.A.* **34** (1948), 204–207.
- [WX16] J. Wu & P. Xi, Arithmetic exponent pairs for algebraic trace functions and applications, arXiv:1603.07060 [math.NT]

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